Cover Times for a Run-and-Tumble Particle in 1D with Stochastic Resetting

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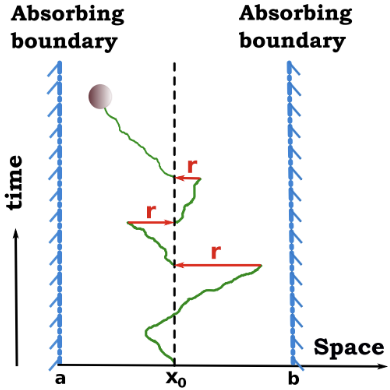
July 17, 2024

Overview of Problem



1

[1]Pal & Prasad

Introduction

A RTP particle's position $x(t)$ evolves per the Langevin equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v \cdot \sigma(t)$$

where

- v is a constant speed
- $\sigma \in \{\pm 1\}$
- $P(\sigma(0) = \pm 1) = 1/2$
- $\sigma \leftarrow -\sigma$ as a Poisson process with rate γ

Introduction

A RTP particle's position $x(t)$ evolves per the Langevin equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v \cdot \sigma(t)$$

where we impose a stochastic resetting:

- The particle spawns from the point $x_0 \in [0, L]$
- The particle resets position to $X_r \in [0, L]$ as a Poisson process with rate r

An Expression for Cover Times

Definition (First Passage Time)

The First Passage Time is the **time it takes** for the particle of dynamics $x(t)$ **to reach a boundary** ω

$$FPT := \inf\{t > 0 : x(t) \rightarrow \omega\}$$

Definition (Cover Time)

The Cover Time of set Ω is the **time it takes** for the particle with dynamics $x(t)$ **to cover every point in** Ω.

$$CT := \inf\{t \geq 0 : \Omega \subseteq \cup_{s=0}^{t} x(t)\}$$

An Expression for Cover Times

The Mean Cover Time can be broken down into three terms as

$$MCT = \mathbb{E}T_{exit} + \pi_0 \mathbb{E}T_{0\to L} + \pi_L \mathbb{E}T_{L\to 0}$$

where

- $\pi_{0/L} = \mathbb{P}(0/L \text{ is reached first})$
- T_{exit} is the time taken for the particle to leave the interval.
- $T_{a\to b}$ is the time taken for a particle spawning at a to reach b.

An Expression for Cover Times

In this work, we assume that the spawn point x_0 and the reset point x_r are both $L/2$. The following simplifications follow:

$$\pi_0 = \pi_L = \frac{1}{2}$$

$$T_{0\to L} = T_{L\to 0} =: T_{halfline}$$

and thus,

$$MCT = \mathbb{E}\,T_{exit} + \mathbb{E}\,T_{halfline}$$

Finding $T_{halfline}$

Definition (Survival Probability)

The survival probability of the particle with resetting, $Q_r(x_0, t)$, is the **probability that the particle has not yet left the interval** at time t, given it spawned at x_0, with resetting rate r.

[2] Pal & Prasad

Finding $T_{halfline}$

Definition (Survival Probability)

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A *Renewal Equation* for $Q_r(x_0, t)$ is [2]

$$Q_r(x_0, t) = \underbrace{e^{-rt} Q_0(x_0, t)}_{\text{contribution from no reset}} + \underbrace{\int_0^t r e^{-rt} Q_0(x_0, \tau) Q_r(x_0, t - \tau) \, \mathrm{d}\tau}_{\text{contribution of reset at time (t - } \tau)}$$

[2] Pal & Prasad

Finding $T_{halfline}$

Consider the Laplace transform

$$\mathcal{L}\{Q_r(x_0,t)\} = \int_0^\infty e^{-st} Q_r(x_0,t)\,dt$$

$$\tilde{Q}_r(x_0,s) = \tilde{Q}_0(x_0,r+s) + r\tilde{Q}_0(x_0,r+s)\tilde{Q}_r(x_0,s)$$

$$\tilde{Q}_r(x_0,s) = \boxed{\frac{\tilde{Q}_0(x_0,r+s)}{1 - r\tilde{Q}_0(x_0,r+s)}}$$

In Laplace space, the survival probability with resetting is neatly related to the survival probability without resetting!

Finding $T_{halfline}$

We can find the $\mathbb{E}T_{halfline}$ as

$$\mathbb{E}T_{halfline} := \int_0^\infty t \cdot P_{T_{halfline}}(t) = \int_0^\infty t \cdot -\frac{\mathrm{d}Q_r(x_0,t)}{\mathrm{d}t} dt$$
$$\overset{\text{IBP}}{=} \int_0^\infty Q_r(x_0,t)dt = \int_0^\infty e^{-0\cdot t} Q_r(x_0,t)dt$$
$$=: \tilde{Q}_r(x_0, s=0)$$
$$= \boxed{\frac{\tilde{Q}_0(x_0,r)}{1 - r\tilde{Q}_0(x_0,r)}}$$

3

[3] Evans and Majumdar, 2018

Expression for $T_{halfline}$

$$\mathbb{E}\, T_{halfine} = \frac{\tilde{Q}_0(x_0, r)}{1 - r\tilde{Q}_0(x_0, r)}$$

$$\tilde{Q}_0(x_0, s) = \frac{1}{s} + \frac{1}{2\gamma s}(v\lambda) - (s + 2\gamma)e^{-\lambda x_0}; \quad \lambda = \sqrt{\frac{s(s+\gamma)}{v^2}}$$

[4]

[4] Evans and Majumdar, 2018

Expression for T_{exit}

$$\mathbb{E}T_{exit} = \frac{-2\gamma - r}{r(2\gamma + r)} + \frac{(2\gamma + r)\cosh\left(\frac{L}{2v}\sqrt{r(2\gamma + r)}\right)}{r(2\gamma + r)} + \frac{v\sqrt{r(2\gamma + r)}\sinh\left(\frac{L}{2v}\sqrt{r(2\gamma + r)}\right)}{r(2\gamma + r)}$$

Putting it together

$$MCT = \frac{\frac{1}{r} + \frac{1}{2\gamma r}(v\sqrt{\frac{r(r+\gamma)}{v^2}}) - (r+2\gamma)e^{-\sqrt{\frac{r(r+\gamma)}{v^2}}x_0}}{1 - r\left(\frac{1}{r} + \frac{1}{2\gamma r}(v\sqrt{\frac{r(r+\gamma)}{v^2}}) - (r+2\gamma)e^{-\sqrt{\frac{r(r+\gamma)}{v^2}}x_0}\right)}$$

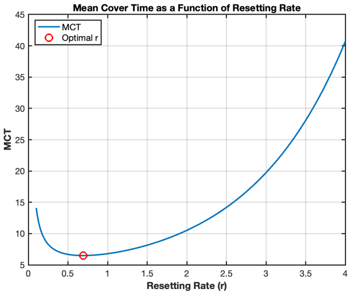
$$+ \frac{-2\gamma - r}{r(2\gamma + r)}$$

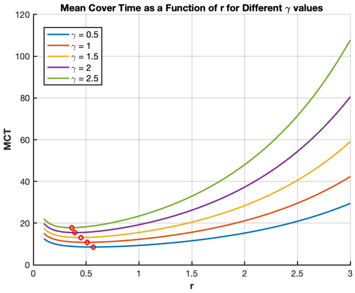
$$+ \frac{(2\gamma + r)\cosh\left(\frac{L}{2v}\sqrt{r(2\gamma + r)}\right)}{r(2\gamma + r)}$$

$$+ \frac{v\sqrt{r(2\gamma + r)}\sinh\left(\frac{L}{2v}\sqrt{r(2\gamma + r)}\right)}{r(2\gamma + r)}$$

Results: Dependence of MCT on resetting rate

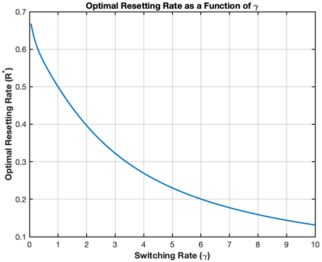
Fixing $v = 1, L = 2$ we plot MCT vs r

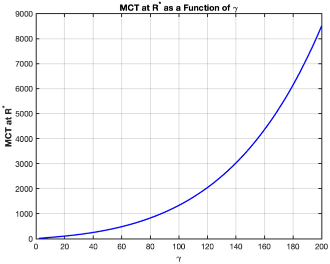




- The MCT has a unique minima for every fixed switching rate γ
- The minimizer r^* shifts closer to zero as $\gamma \uparrow$

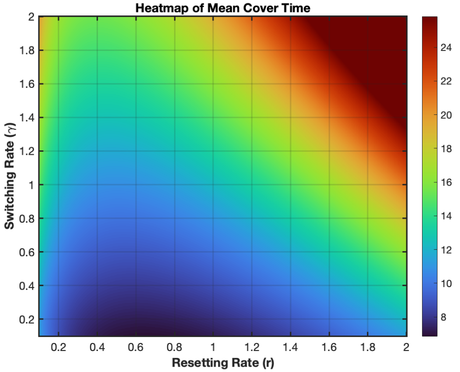
Results: Dependence of optimal resetting rate, MCT on switching rate



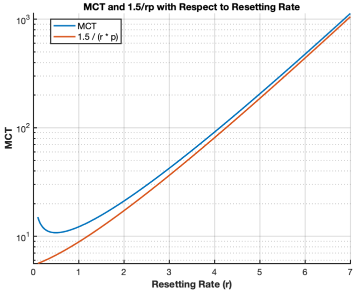


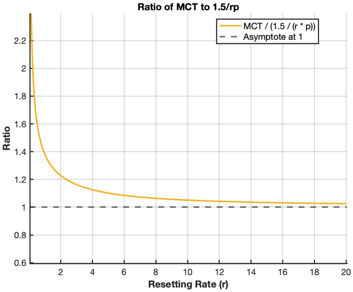
Here the decay and growth are both subexponential.

Results: Dependence of MCT on switching and resetting rate



Results: Large r (freq) behavior of MCT





- $MCT \sim \frac{3/2}{rp}$ as $r \to \infty$ in agreement with the analysis of Linn & Lawley (2024) for Brownian Motion on a symmetric interval.
- p is the measure of the success probability (of reaching a boundary before resetting).

Future Research

- Velocity is not restricted to one value
- Different distributions of resetting and switching
- Extending this to higher dimensions, where the particle is equipped with a detection radius.

References

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Acknowledgements

Special Thanks to Sean Lawley and Samantha Linn

for organizing and assisting this program!